

Random Gaussian Tetrahedra

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ABSTRACT. Given independent normally distributed points A, B, C, D in Euclidean 3-space, let Q denote the plane determined by A, B, C and \tilde{D} denote the orthogonal projection of D onto Q . The probability that the tetrahedron $ABCD$ is acute remains intractible. We make some small progress in resolving this issue. Let Γ denote the convex cone in Q containing all linear combinations $A + r(B - A) + s(C - A)$ for nonnegative r, s . We compute the probability that \tilde{D} falls in $(B + C) - \Gamma$ to be 0.681..., but the probability that \tilde{D} falls in Γ to be 0.683.... The intersection of these two cones is a parallelogram in Q twice the area of the triangle ABC . Among other issues, we mention the distribution of random solid angles and sums of these.

Let $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3, d_1, d_2, d_3$ be independent normally distributed random variables with mean 0 and variance 1. The points

$$A = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \quad C = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}, \quad D = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

constitute the vertices of a tetrahedron in Euclidean 3-space. The tetrahedron $ABCD$ is **acute** if each of its six internal dihedral angles are less than $\pi/2$. It is known [1] that $ABCD$ is acute if and only if the orthogonal projection of each vertex onto the plane of the opposite face lies within that face. Neither characterization suggests an easy approach to finding the probability that random $ABCD$ is acute. We will examine a variation of the latter characterization, focusing on the point D . Let Q denote the plane determined by A, B, C and \tilde{D} denote the orthogonal projection of D onto Q . While calculating the probability that \tilde{D} falls within the triangle ABC seems difficult, we succeed in computing the probability that \tilde{D} falls in either of two convex cones in Q containing ABC . In fact, the cones contain the parallelogram with vertices $A, B, C, -A + B + C$; the fourth vertex is clearly the vector sum $(B - A) + (C - A)$ displaced so that it emanates from the point A . Moreover, the intersection of the cones gives precisely the parallelogram. Further research might uncover other probabilities for related cones in Q , and thus the inclusion-exclusion principle might yield the

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probability that \tilde{D} falls within the parallelogram. This task is best left to someone else (!) but we hope that our work provides some inspiration along the way.

In section 1, we review a well-known proof that planar triangles ABC are acute with probability $1/4$. Returning to 3-space, we compute in section 2 the probability that \tilde{D} falls in the cone

$$(-A + B + C) - \{r(B - A) + s(C - A) : r \geq 0, s \geq 0\}$$

using the Krishnaiah bivariate F -ratio distribution [2, 3, 4]. This is the cone with vertex $-A + B + C$ and outgoing edges parallel to vectors $A - B$, $A - C$. In the following section, we compute the probability that \tilde{D} falls in the cone

$$\Gamma = A + \{r(B - A) + s(C - A) : r \geq 0, s \geq 0\}$$

using a forgotten result of Miller's [5], extended via convolution. This is the cone with vertex A and outgoing edges parallel to vectors $B - A$, $C - A$. The two probabilities are numerically close but not equal (at least not in our view). A rigorous proof of such an inequality is open.

1. TRIANGLES AND TETRAHEDRA

For the moment,

$$A = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \quad C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

are vertices of a random Gaussian triangle in Euclidean 2-space. Let Q denote the line determined by A, B and \tilde{C} be analogous to before. Also, let α, β, γ denote the angles at A, B, C respectively. The orthogonal projection of $\vec{w} = C - A$ onto the subspace spanned by $\vec{v} = B - A$ is

$$\frac{\vec{v} \cdot \vec{w}}{\vec{v} \cdot \vec{v}} \vec{v} = \frac{(B - A) \cdot (C - A)}{(B - A) \cdot (B - A)} (B - A)$$

and clearly \tilde{C} falls between A and B on Q if and only if

$$0 < \frac{(B - A) \cdot (C - A)}{(B - A) \cdot (B - A)} < 1.$$

Let

$$p = \text{P}((B - A) \cdot (C - A) > 0) = \text{P}\left(\frac{(B - A) \cdot (C - A)}{(B - A) \cdot (B - A)} > 0\right),$$

then

$$\begin{aligned}
\mathrm{P}\left(\frac{(B-A) \cdot (C-A)}{(B-A) \cdot (B-A)} < 1\right) &= \mathrm{P}((B-A) \cdot (C-A) < (B-A) \cdot (B-A)) \\
&= \mathrm{P}((B-A) \cdot [(C-A) - (B-A)] < 0) \\
&= \mathrm{P}((B-A) \cdot (C-B) < 0) \\
&= \mathrm{P}((A-B) \cdot (C-B) > 0) = p
\end{aligned}$$

by symmetry, hence

$$\mathrm{P}\left(0 < \frac{(B-A) \cdot (C-A)}{(B-A) \cdot (B-A)} < 1\right) = p - (1-p) = 2p - 1.$$

To compute p , note that $(B-A) \cdot (C-A)$ is equal to

$$\begin{aligned}
&\frac{3}{2} \left(-\sqrt{\frac{2}{3}}A + \frac{B}{\sqrt{6}} + \frac{C}{\sqrt{6}} \right) \cdot \left(-\sqrt{\frac{2}{3}}A + \frac{B}{\sqrt{6}} + \frac{C}{\sqrt{6}} \right) - \frac{1}{2} \left(-\frac{B}{\sqrt{2}} + \frac{C}{\sqrt{2}} \right) \cdot \left(-\frac{B}{\sqrt{2}} + \frac{C}{\sqrt{2}} \right) \\
&= \frac{3}{2} \left\| -\sqrt{\frac{2}{3}}A + \frac{B}{\sqrt{6}} + \frac{C}{\sqrt{6}} \right\|^2 - \frac{1}{2} \left\| -\frac{B}{\sqrt{2}} + \frac{C}{\sqrt{2}} \right\|^2 \\
&= \frac{3}{2} \chi_2^2 - \frac{1}{2} \bar{\chi}_2^2,
\end{aligned}$$

a linear combination of independent chi-square distributed variables (each with 2 degrees of freedom). Therefore

$$p = \mathrm{P}\left(\frac{3}{2} \chi_2^2 - \frac{1}{2} \bar{\chi}_2^2 > 0\right) = \mathrm{P}\left(\frac{\chi_2^2}{\bar{\chi}_2^2} > \frac{1}{3}\right) = \mathrm{P}\left(F_{2,2} > \frac{1}{3}\right) = \frac{3}{4}$$

by use of the F -ratio distribution with $(2, 2)$ degrees of freedom. It follows that \tilde{C} falls between A and B on Q with probability $2(3/4) - 1 = 1/2$. Equivalently, $\mathrm{P}(\alpha > \pi/2) = 1/4$ because $\cos(\alpha) < 0$ precisely when $(B-A) \cdot (C-A) < 0$. Since at most one angle of a triangle can be obtuse, we deduce that

$$\begin{aligned}
\mathrm{P}(\text{triangle } ABC \text{ is acute}) &= 1 - \mathrm{P}(\text{triangle } ABC \text{ is obtuse}) \\
&= 1 - (\mathrm{P}(\alpha > \pi/2) + \mathrm{P}(\beta > \pi/2) + \mathrm{P}(\gamma > \pi/2)) \\
&= 1 - 3/4 = 1/4
\end{aligned}$$

as was to be shown. Our proof imitates Eisenberg & Sullivan's [6] approach, although we avoid angles α, β, γ until the last step, using \tilde{C} instead. Portnoy's [7] argument employs triangle medians rather than orthogonal projections.

Let us now return to random Gaussian tetrahedra in 3-space. Our argument is similar but more complicated. The orthogonal projection of $\vec{w} = D - A$ onto the subspace spanned by $\vec{u} = B - A$, $\vec{v} = C - A$ is

$$\frac{\vec{u} \cdot \vec{w}}{\vec{u} \cdot \vec{u}} \vec{u} + \frac{\vec{v} \cdot \vec{w}}{\vec{v} \cdot \vec{v}} \vec{v} = \frac{(B - A) \cdot (D - A)}{(B - A) \cdot (B - A)} (B - A) + \frac{(C - A) \cdot (D - A)}{(C - A) \cdot (C - A)} (C - A)$$

and clearly \tilde{D} falls within the desired parallelogram on Q if and only if

$$0 < \frac{(B - A) \cdot (D - A)}{(B - A) \cdot (B - A)} < 1 \quad \text{and} \quad 0 < \frac{(C - A) \cdot (D - A)}{(C - A) \cdot (C - A)} < 1.$$

This is the same as requiring that

$$\begin{aligned} (A - B) \cdot (D - B) &> 0, & (B - A) \cdot (D - A) &> 0, \\ (A - C) \cdot (D - C) &> 0, & (C - A) \cdot (D - A) &> 0. \end{aligned}$$

Each product is of the form $(3/2)\chi_3^2 - (1/2)\bar{\chi}_3^2$ and jointly they give rise to corresponding $F_{3,3}$ ratios:

$$\begin{aligned} \frac{\left\| \frac{A}{\sqrt{6}} - \sqrt{\frac{2}{3}}B + \frac{D}{\sqrt{6}} \right\|^2}{\left\| -\frac{A}{\sqrt{2}} + \frac{D}{\sqrt{2}} \right\|^2} &> \frac{1}{3}, & \frac{\left\| -\sqrt{\frac{2}{3}}A + \frac{B}{\sqrt{6}} + \frac{D}{\sqrt{6}} \right\|^2}{\left\| -\frac{B}{\sqrt{2}} + \frac{D}{\sqrt{2}} \right\|^2} &> \frac{1}{3}, \\ \frac{\left\| \frac{A}{\sqrt{6}} - \sqrt{\frac{2}{3}}C + \frac{D}{\sqrt{6}} \right\|^2}{\left\| -\frac{A}{\sqrt{2}} + \frac{D}{\sqrt{2}} \right\|^2} &> \frac{1}{3}, & \frac{\left\| -\sqrt{\frac{2}{3}}A + \frac{C}{\sqrt{6}} + \frac{D}{\sqrt{6}} \right\|^2}{\left\| -\frac{C}{\sqrt{2}} + \frac{D}{\sqrt{2}} \right\|^2} &> \frac{1}{3}. \end{aligned}$$

A computation of the probability that all four inequalities hold simultaneously does not seem possible. We note that the two expressions in the left-hand column possess the same denominator, which is essential for the next section. An entirely different approach will be needed for the two expressions in the right-hand column. Finally, the two left-hand inequalities are true if and only if the orthogonal projection of \vec{w} is $r\vec{u} + s\vec{v}$ for some $r \leq 1$, $s \leq 1$; this translates into \tilde{D} falling in the cone

$$A + \{(1 - r)(B - A) + (1 - s)(C - A) : r \geq 0, s \geq 0\} = (B + C) - \Gamma.$$

Likewise, the two right-hand inequalities are true if and only if the orthogonal projection of \vec{w} is $r\vec{u} + s\vec{v}$ for some $r \geq 0$, $s \geq 0$; this translates immediately into \tilde{D} falling in the cone Γ .

2. KRISHNAIAH BIVARIATE F -RATIO DISTRIBUTION

Let X be an $n \times 2$ matrix of n independent random row 2-vectors, each distributed according to $N(0, \Sigma)$, where

$$\Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}, \quad -1 < \rho < 1.$$

Let $\hat{\sigma}_{ij} = X_i \cdot X_j$, where X_k is the k^{th} column of X for $k = 1, 2$. Also let $\hat{\tau} = Y \cdot Y$, where Y is a random column m -vector satisfying $Y \sim N(0, I)$ and independent of X . Then the joint distribution of $(m \hat{\sigma}_{11})/(n \hat{\tau})$, $(m \hat{\sigma}_{22})/(n \hat{\tau})$ is a bivariate F -ratio distribution with (n, m) degrees of freedom and with Σ as the associated covariance matrix (for X). These conditions are clearly met in our case, for which $n = m = 3$,

$$X = \begin{pmatrix} \frac{a_1}{\sqrt{6}} - \sqrt{\frac{2}{3}}b_1 + \frac{d_1}{\sqrt{6}} & \frac{a_1}{\sqrt{6}} - \sqrt{\frac{2}{3}}c_1 + \frac{d_1}{\sqrt{6}} \\ \frac{a_2}{\sqrt{6}} - \sqrt{\frac{2}{3}}b_2 + \frac{d_2}{\sqrt{6}} & \frac{a_2}{\sqrt{6}} - \sqrt{\frac{2}{3}}c_2 + \frac{d_2}{\sqrt{6}} \\ \frac{a_3}{\sqrt{6}} - \sqrt{\frac{2}{3}}b_3 + \frac{d_3}{\sqrt{6}} & \frac{a_3}{\sqrt{6}} - \sqrt{\frac{2}{3}}c_3 + \frac{d_3}{\sqrt{6}} \end{pmatrix}, \quad Y = \begin{pmatrix} -\frac{a_1}{\sqrt{2}} + \frac{d_1}{\sqrt{2}} \\ -\frac{a_2}{\sqrt{2}} + \frac{d_2}{\sqrt{2}} \\ -\frac{a_3}{\sqrt{2}} + \frac{d_3}{\sqrt{2}} \end{pmatrix}$$

and

$$\begin{aligned} \rho &= \text{Cov} \left(\frac{a_k}{\sqrt{6}} - \sqrt{\frac{2}{3}}b_k + \frac{d_k}{\sqrt{6}}, \frac{a_k}{\sqrt{6}} - \sqrt{\frac{2}{3}}c_k + \frac{d_k}{\sqrt{6}} \right) \\ &= \text{Var} \left(\frac{a_k}{\sqrt{6}} + \frac{d_k}{\sqrt{6}} \right) = \frac{1}{3}. \end{aligned}$$

It follows that [3]

$$P \left(\frac{m \hat{\sigma}_{11}}{n \hat{\tau}} > \xi \text{ and } \frac{m \hat{\sigma}_{22}}{n \hat{\tau}} > \xi \right) = \frac{(1 - \rho^2)^{n/2}}{\Gamma(m/2)\Gamma(n/2)} \sum_{k=0}^{\infty} \frac{\rho^{2k} \Gamma(n + m/2 + 2k)}{k! \Gamma(n/2 + k)} \Lambda_k$$

where

$$\Lambda_k = \int_{\eta}^{\infty} \int_{\eta}^{\infty} \frac{(xy)^{n/2+k-1}}{(1+x+y)^{n+2k+m/2}} dx dy.$$

and $\eta = (n\xi)/(m(1 - \rho^2))$. Each Λ_k can be evaluated symbolically and the series appears to converge fairly quickly. For our case, $\xi = 1/3$, hence $\eta = 3/8$ and the desired probability is 0.6810669069....

3. MILLER BIVARIATE DENSITY FOR PRODUCTS

We want the probability that both $(B - A) \cdot (D - A) > 0$ and $(C - A) \cdot (D - A) > 0$. For now, we consider a simpler scenario in which $a = A, b = B, c = C, d = D$ are scalars.

Let X be a random p -vector and y be a random scalar. Assume that $(X, y) \sim N(0, \Sigma)$ and that the $(p + 1) \times (p + 1)$ covariance matrix Σ has inverse

$$\Sigma^{-1} = \begin{pmatrix} \Omega & v \\ v' & \omega \end{pmatrix}$$

where Ω is $p \times p$, v is $p \times 1$ and ω is a scalar. Let $Z = yX$. Then the joint density of Z is [5]

$$\frac{2\sqrt{\det(\Sigma^{-1})}}{(2\pi)^{(p+1)/2}} \left(\frac{\omega}{Z'\Omega Z} \right)^{(p-1)/4} \exp(-v'Z) K_{(p-1)/2}(\sqrt{\omega Z'\Omega Z})$$

where $K_{(p-1)/2}(\theta)$ is the modified Bessel function of the second kind. In our case, $p = 2$,

$$\Sigma = \text{Cov} \left(\begin{pmatrix} x_1 \\ x_2 \\ y \end{pmatrix} \right) = \text{Cov} \left(\begin{pmatrix} b - a \\ c - a \\ d - a \end{pmatrix} \right) = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

hence

$$\Sigma^{-1} = \begin{pmatrix} 3/4 & -1/4 & -1/4 \\ -1/4 & 3/4 & -1/4 \\ -1/4 & -1/4 & 3/4 \end{pmatrix}, \quad \Omega = \frac{1}{4} \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}, \quad v = -\frac{1}{4} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \omega = \frac{3}{4},$$

$$\sqrt{\det(\Sigma^{-1})} = \frac{1}{2}, \quad -v'Z = \frac{1}{4}(z_1 + z_2), \quad Z'\Omega Z = \frac{1}{4}(3z_1^2 - 2z_1z_2 + 3z_2^2).$$

Also

$$K_{1/2}(\theta) = \sqrt{\frac{\pi}{2}} \frac{\exp(-\theta)}{\sqrt{\theta}}$$

thus the density simplifies to

$$f(z_1, z_2) = \frac{1}{2\pi} \frac{\exp\left(\frac{1}{4}\left(z_1 + z_2 - \sqrt{3}\sqrt{3z_1^2 - 2z_1z_2 + 3z_2^2}\right)\right)}{\sqrt{3z_1^2 - 2z_1z_2 + 3z_2^2}}.$$

In fact, we wish to compute the probability that the sum of *three* independent copies of Z has both components > 0 . One way to do this is to evaluate the sextuple integral:

$$\int_0^\infty \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty f(z_1 - u_1 - v_1, z_2 - u_2 - v_2) f(u_1, u_2) f(v_1, v_2) du_1 du_2 dv_1 dv_2 dz_1 dz_2 = 0.683...$$

obtained via repeated convolution of f with itself. Another way uses the Fourier transform

$$F(w_1, w_2) = \frac{1}{\sqrt{(w_1 - i)(3w_1 + i) + (w_2 - i)(3w_2 + i) + 2w_1w_2 - 1}}$$

which, when cubed, maps back to a remarkably simple density function. Higher precision is now possible:

$$\int_0^\infty \int_0^\infty \frac{1}{4\sqrt{3}\pi} \exp\left(\frac{1}{4}\left(z_1 + z_2 - \sqrt{3}\sqrt{3z_1^2 - 2z_1z_2 + 3z_2^2}\right)\right) dz_1 dz_2 = 0.6837762984\dots$$

and the elaborate details appear later.

It is surprising that working with $(B-A) \cdot (D-A)$, $(C-A) \cdot (D-A)$ simultaneously should be so difficult. The two expressions are familiar: they are sample covariance coefficients $\hat{\gamma}_{13}$, $\hat{\gamma}_{23}$ respectively between samples of size $= 3$. A marginal density for either is found in [8], but not much else is known about off-diagonal elements of a Wishart matrix [9]. Of course, $\hat{\gamma}_{13} > 0$ and $\hat{\gamma}_{23} > 0$ if and only if the corresponding sample correlation coefficients $\hat{\rho}_{13} > 0$ and $\hat{\rho}_{23} > 0$. A formula for a trivariate density for $(\hat{\rho}_{12}, \hat{\rho}_{13}, \hat{\rho}_{23})$ is outlined in [10, 11] – evidently a sample size > 4 is presumed – and details still need to come together.

4. PINNED SIMPLICES

A slight variation on defining a random tetrahedron in 3-space is to keep one vertex fixed at the origin and to select the other three vertices independently from $N(0, I)$ as before. We say that the tetrahedron is **pinned**.

For the moment, let us consider pinned random Gaussian triangles ABC in 2-space with $C = (0, 0)$. Note that $(B - A) \cdot (-A)$ is equal to [6]

$$\begin{aligned} & \frac{1 + \sqrt{2}}{2} \left\| -\frac{\sqrt{2 + \sqrt{2}}}{2} A + \frac{\sqrt{2 - \sqrt{2}}}{2} B \right\|^2 - \frac{-1 + \sqrt{2}}{2} \left\| \frac{\sqrt{2 - \sqrt{2}}}{2} A + \frac{\sqrt{2 + \sqrt{2}}}{2} B \right\|^2 \\ &= \frac{1 + \sqrt{2}}{2} \chi_2^2 - \frac{-1 + \sqrt{2}}{2} \bar{\chi}_2^2, \end{aligned}$$

therefore

$$p = \mathbb{P}\left(\frac{1 + \sqrt{2}}{2} \chi_2^2 - \frac{-1 + \sqrt{2}}{2} \bar{\chi}_2^2 > 0\right) = \mathbb{P}\left(F_{2,2} > 3 - 2\sqrt{2}\right) = \frac{2 + \sqrt{2}}{4}.$$

It follows that \tilde{C} falls between A and B on Q with probability $2p - 1 = 1/\sqrt{2}$. Equivalently, $\mathbb{P}(\alpha > \pi/2) = 1 - p$ because $\cos(\alpha) < 0$ if and only if $(B - A) \cdot (-A) < 0$.

By symmetry, $P(\beta > \pi/2) = 1 - p$ as well and $P(\gamma > \pi/2) = 1/2$. We deduce that

$$\begin{aligned} & P(\text{pinned triangle } ABC \text{ is acute}) \\ &= 1 - (P(\alpha > \pi/2) + P(\beta > \pi/2) + P(\gamma > \pi/2)) \\ &= 1 - (2 - 2p + 1/2) = -1/2 + 1/\sqrt{2} \end{aligned}$$

as was to be shown.

Let us now return to pinned random Gaussian tetrahedra in 3-space with $D = (0, 0, 0)$. For brevity, we focus only on the probability that both $(B - A) \cdot (-A) > 0$ and $(C - A) \cdot (-A) > 0$. Consider a simpler scenario in which $a = A, b = B, c = C$ are scalars. Using Miller's [5] formulas,

$$\Sigma = \text{Cov} \left(\begin{pmatrix} b - a \\ c - a \\ -a \end{pmatrix} \right) = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

hence

$$\Sigma^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 3 \end{pmatrix}, \quad \Omega = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad v = - \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \omega = 3,$$

$$\sqrt{\det(\Sigma^{-1})} = 1, \quad -v'Z = z_1 + z_2, \quad Z'\Omega Z = z_1^2 + z_2^2,$$

$$f(z_1, z_2) = \frac{1}{2\pi} \frac{\exp \left(z_1 + z_2 - \sqrt{3} \sqrt{z_1^2 + z_2^2} \right)}{\sqrt{z_1^2 + z_2^2}}.$$

The probability we want is given by the sextuple integral, which has value 0.834..., but can be computed more accurately via the Fourier transform

$$F(w_1, w_2) = \frac{1}{\sqrt{(w_1 - i)^2 + (w_2 - i)^2 + 3}}.$$

Mapping $F(w_1, w_2)^3$ from frequency back to signal domain, we calculate

$$\int_0^\infty \int_0^\infty \frac{1}{2\sqrt{3}\pi} \exp \left(z_1 + z_2 - \sqrt{3} \sqrt{z_1^2 + z_2^2} \right) dz_1 dz_2 = 0.8343764256....$$

After a discussion of some related problems in geometric probability [12], more information on $F(w_1, w_2)$ (also called the *characteristic function* corresponding to $f(z_1, z_2)$) will be given.

5. RANDOM SOLID ANGLES

We restrict attention to pinned Gaussian random tetrahedra $ABCD$ with $D = (0, 0, 0)$. The **dihedral angle** α is the angle between normal vectors $A \times B$, $A \times C$ to the triangular faces ADB , ADC respectively:

$$\alpha = \arccos \left(\frac{(A \times B) \cdot (A \times C)}{\|A \times B\| \|A \times C\|} \right).$$

Angles β and γ are defined likewise. For example, a *regular* tetrahedron has dihedral angles each equal to

$$\arccos(1/3) = 1.2309594173... = \pi - 1.9106332362... \approx 70.53^\circ \approx 180^\circ - 109.47^\circ.$$

The joint density for α, β, γ is [13]

$$\begin{cases} -\frac{1}{\pi} \frac{\cos\left(\frac{x+y+z}{2}\right) \cos\left(\frac{-x+y+z}{2}\right) \cos\left(\frac{x-y+z}{2}\right) \cos\left(\frac{x+y-z}{2}\right)}{\sin(x)^2 \sin(y)^2 \sin(z)^2} & \text{if } x+y+z > \pi, x+y < \pi+z, y+z < \pi+x \text{ and } z+x < \pi+y, \\ 0 & \text{otherwise.} \end{cases}$$

As a consequence, α is uniformly distributed on $[0, \pi]$ and angles α, β, γ are uncorrelated but pairwise dependent ((α, β) is not uniform on $[0, \pi] \times [0, \pi]$).

The **solid angle** (or **trihedral angle**) at D is the area of the spherical triangle on the unit sphere, center D , with vertices $A/\|A\|$, $B/\|B\|$, $C/\|C\|$. It is equal to the spherical excess $\alpha + \beta + \gamma - \pi$, which is between 0 and 2π . It is also equal to [14, 15]

$$\begin{cases} 2 \arctan(\zeta) & \text{if } \zeta \geq 0, \\ 2\pi + 2 \arctan(\zeta) & \text{if } \zeta < 0 \end{cases}$$

where

$$\zeta = \frac{|A \cdot (B \times C)|}{\|A\| \|B\| \|C\| + (A \cdot B) \|C\| + (A \cdot C) \|B\| + (B \cdot C) \|A\|}.$$

It can be shown that, if a tetrahedron is acute, then each of its four solid angles are less than $\pi/2$, but not conversely [16]. A proposed density for the solid angle at D was published in 1867 [17]:

$$-\frac{(x^2 - 4\pi x + 3\pi^2 - 6) \cos(x) - 6(x - 2\pi) \sin(x) - 2(x^2 - 4\pi x + 3\pi^2 + 3)}{16\pi \cos(x/2)^4}$$

for $0 < x < 2\pi$ and remained obscure until it was cited in a recent paper [18]. Details of the supporting geometric proof need to be carefully examined. No analytic proof using the joint density for α, β, γ has yet been found.

As far as is known, no analogous results are known for general Gaussian random tetrahedra. In particular, the sum σ of the four solid angles associated with a tetrahedra T possesses a fascinating property [19]:

$$\frac{\sigma}{2\pi} = \mathbb{P} \left(\begin{array}{c} \text{the orthogonal projection of } T \text{ onto a uniform} \\ \text{random plane in 3-space is a triangle} \end{array} \right)$$

and it would be good to understand σ more fully. As an example, the regular tetrahedra has solid angles each equal to

$$3 \arccos(1/3) - \pi = 0.5512855984\dots, \quad \text{hence } \sigma/(2\pi) = 0.3509593121\dots$$

and this is the maximum such value over all *equifacial* tetrahedra (all faces are congruent) [20, 21, 22, 23]. No one has studied the distribution of σ when T is itself allowed to be random.

While we know the mean volume of a tetrahedron [24, 25, 26, 27] with uniform random vertices in the unit ball ($12\pi/715$) and with uniform random vertices in the unit cube ($3977/21600 - \pi^2/2160$), the Gaussian random scenario remains open (there is doubt about claims in [28]).

In 2-space, a triangle is acute if and only if its circumcenter lies inside the triangle (the circumcircle contains all three vertices). In 3-space, a tetrahedron is **3-well-centered** if its circumcenter lies inside the tetrahedron; a tetrahedron is **2-well-centered** if the circumcenter of each face lies inside the face. An acute T can fail to be 3-well-centered, and a 3-well-centered T can fail to be acute. However, an acute T must be 2-well-centered (equivalently, all its faces must be acute) but not conversely [1, 29]. Many problems involving Gaussian random tetrahedra suggest themselves.

6. FOURIER TRANSFORMS

It is not difficult to prove all the formulas we need in the “forward” direction (starting with f and ending with F). This is done first for the pinned case, which is easier, and then for the general case. Motivating the formula for the inverse Fourier transform of F^3 is harder. We do this for the pinned case only.

6.1. Pinned Case: Forward Direction. Our objective is to evaluate two integrals:

$$F(u, v) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\exp \left((1 + iu)x + (1 + iv)y - \sqrt{3}\sqrt{x^2 + y^2} \right)}{\sqrt{x^2 + y^2}} dx dy,$$

$$G(u, v) = \frac{1}{2\sqrt{3}\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \left((1 + iu)x + (1 + iv)y - \sqrt{3}\sqrt{x^2 + y^2} \right) dx dy.$$

Let $x = r \cos(\theta)$, $y = r \sin(\theta)$, then $dx dy = r dr d\theta$ and

$$\begin{aligned}
F(u, v) &= \frac{1}{2\pi} \int_0^{2\pi} \int_0^\infty \exp \left(r \left[(1 + i u) \cos(\theta) + (1 + i v) \sin(\theta) - \sqrt{3} \right] \right) dr d\theta \\
&= -\frac{1}{2\pi} \int_0^{2\pi} \frac{1}{(1 + i u) \cos(\theta) + (1 + i v) \sin(\theta) - \sqrt{3}} d\theta, \\
G(u, v) &= \frac{1}{2\sqrt{3}\pi} \int_0^{2\pi} \int_0^\infty r \exp \left(r \left[(1 + i u) \cos(\theta) + (1 + i v) \sin(\theta) - \sqrt{3} \right] \right) dr d\theta \\
&= \frac{1}{2\sqrt{3}\pi} \int_0^{2\pi} \frac{1}{\left[(1 + i u) \cos(\theta) + (1 + i v) \sin(\theta) - \sqrt{3} \right]^2} d\theta
\end{aligned}$$

since

$$\begin{aligned}
\operatorname{Re} \left[(1 + i u) \cos(\theta) + (1 + i v) \sin(\theta) - \sqrt{3} \right] &= \cos(\theta) + \sin(\theta) - \sqrt{3} \\
&\leq \sqrt{2} - \sqrt{3} < 0.
\end{aligned}$$

Let $z = \exp(i\theta)$, then $d\theta = -i dz/z$ and

$$\begin{aligned}
F(u, v) &= \frac{i}{2\pi} \int_C \frac{1}{(1 + i u) \frac{1}{2} \left(z + \frac{1}{z} \right) + (1 + i v) \frac{1}{2i} \left(z - \frac{1}{z} \right) - \sqrt{3}} \frac{dz}{z} \\
&= \frac{i}{\pi} \int_C \frac{1}{(1 + i u)(z^2 + 1) - i(1 + i v)(z^2 - 1) - 2\sqrt{3}z} dz \\
&= \frac{i}{\pi} \int_C \frac{1}{(v + 1 + i u - i)z^2 - 2\sqrt{3}z - (v - 1 - i u - i)} dz, \\
G(u, v) &= -\frac{i}{2\sqrt{3}\pi} \int_C \frac{1}{\left[(1 + i u) \frac{1}{2} \left(z + \frac{1}{z} \right) + (1 + i v) \frac{1}{2i} \left(z - \frac{1}{z} \right) - \sqrt{3} \right]^2} \frac{dz}{z} \\
&= -\frac{2i}{\sqrt{3}\pi} \int_C \frac{z}{\left[(v + 1 + i u - i)z^2 - 2\sqrt{3}z - (v - 1 - i u - i) \right]^2} dz
\end{aligned}$$

where C denotes the unit circle, center 0, in the complex plane. The two poles z_{pos} , z_{neg} of each integrand are

$$\frac{2\sqrt{3} \pm \sqrt{12 + 4(v + 1 + i u - i)(v - 1 - i u - i)}}{2(v + 1 + i u - i)} = \frac{\sqrt{3} \pm \sqrt{(u - i)^2 + (v - i)^2 + 3}}{v + 1 + i u - i}$$

and z_{neg} is always inside C , z_{pos} is always outside. For F , z_{neg} is a pole of order 1 and the associated residue is

$$\lim_{z \rightarrow z_{\text{neg}}} \frac{1}{(v+1+iu-i)(z-z_{\text{pos}})} = -\frac{1}{2} \frac{1}{\sqrt{(u-i)^2 + (v-i)^2 + 3}};$$

multiplying by $(2\pi i)(i/\pi)$ completes the proof. For G , z_{neg} is a pole of order 2 and the associated residue is

$$\lim_{z \rightarrow z_{\text{neg}}} \frac{d}{dz} \left\{ \frac{z}{(v+1+iu-i)^2(z-z_{\text{pos}})^2} \right\} = \frac{\sqrt{3}}{4} \frac{1}{[(u-i)^2 + (v-i)^2 + 3]^{3/2}};$$

multiplying by $(2\pi i)(-2i/(\sqrt{3}\pi))$ completes the proof.

6.2. General Case: Forward Direction. Our objective is to evaluate two integrals:

$$F(u, v) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\exp\left(\frac{1}{4} \left((1+4iu)x + (1+4iv)y - \sqrt{3}\sqrt{3x^2 - 2xy + 3y^2} \right)\right)}{\sqrt{3x^2 - 2xy + 3y^2}} dx dy,$$

$$G(u, v) = \frac{1}{4\sqrt{3}\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left(\frac{1}{4} \left((1+4iu)x + (1+4iv)y - \sqrt{3}\sqrt{3x^2 - 2xy + 3y^2} \right)\right) dx dy.$$

Let

$$x = \frac{r}{2\sqrt{2}} (\sqrt{2}\cos(\theta) - \sin(\theta)), \quad y = \frac{r}{2\sqrt{2}} (\sqrt{2}\cos(\theta) + \sin(\theta))$$

then $3x^2 - 2xy + 3y^2 = r^2$ and the Jacobian determinant is

$$\left(\frac{1}{2\sqrt{2}} \right)^2 \begin{vmatrix} \sqrt{2}\cos(\theta) - \sin(\theta) & r(-\sqrt{2}\sin(\theta) - \cos(\theta)) \\ \sqrt{2}\cos(\theta) + \sin(\theta) & r(-\sqrt{2}\sin(\theta) + \cos(\theta)) \end{vmatrix} = \frac{r}{2\sqrt{2}}$$

hence $dx dy = (r/(2\sqrt{2})) dr d\theta$. We obtain

$$\begin{aligned} F(u, v) &= \frac{1}{2\pi} \frac{1}{2\sqrt{2}} \int_0^{2\pi} \int_0^{\infty} \exp\left(\frac{1}{4} \frac{r}{2\sqrt{2}} \left[(1+4iu)(\sqrt{2}\cos(\theta) - \sin(\theta)) \right. \right. \\ &\quad \left. \left. + (1+4iv)(\sqrt{2}\cos(\theta) + \sin(\theta)) - 2\sqrt{2} \cdot \sqrt{3} \right] \right) dr d\theta \\ &= \frac{1}{4\sqrt{2}\pi} \int_0^{2\pi} \int_0^{\infty} \exp\left(\frac{r}{4} \left[\cos(\theta) + 2i(u+v)\cos(\theta) - \sqrt{2}i(u-v)\sin(\theta) - \sqrt{3} \right] \right) dr d\theta \\ &= -\frac{1}{\sqrt{2}\pi} \int_0^{2\pi} \frac{1}{\cos(\theta) + 2i(u+v)\cos(\theta) - \sqrt{2}i(u-v)\sin(\theta) - \sqrt{3}} d\theta, \end{aligned}$$

$$\begin{aligned}
G(u, v) &= \frac{1}{4\sqrt{3}\pi} \frac{1}{2\sqrt{2}} \int_0^{2\pi} \int_0^\infty r \exp \left(\frac{1}{4} \frac{r}{2\sqrt{2}} \left[(1 + 4i u) (\sqrt{2} \cos(\theta) - \sin(\theta)) \right. \right. \\
&\quad \left. \left. + (1 + 4i v) (\sqrt{2} \cos(\theta) + \sin(\theta)) - 2\sqrt{2} \cdot \sqrt{3} \right] \right) dr d\theta \\
&= \frac{1}{8\sqrt{6}\pi} \int_0^{2\pi} \int_0^\infty r \exp \left(\frac{r}{4} \left[\cos(\theta) + 2i(u + v) \cos(\theta) - \sqrt{2}i(u - v) \sin(\theta) - \sqrt{3} \right] \right) dr d\theta \\
&= \frac{2}{\sqrt{6}\pi} \int_0^{2\pi} \frac{1}{\left[\cos(\theta) + 2i(u + v) \cos(\theta) - \sqrt{2}i(u - v) \sin(\theta) - \sqrt{3} \right]^2} d\theta
\end{aligned}$$

since

$$\begin{aligned}
\operatorname{Re} \left[\cos(\theta) + 2i(u + v) \cos(\theta) - \sqrt{2}i(u - v) \sin(\theta) - \sqrt{3} \right] &= \cos(\theta) - \sqrt{3} \\
&\leq 1 - \sqrt{3} < 0.
\end{aligned}$$

Let $z = \exp(i\theta)$, then $d\theta = -i dz/z$ and

$$\begin{aligned}
F(u, v) &= \frac{i}{\sqrt{2}\pi} \int_C \frac{1}{\frac{1}{2}(z + \frac{1}{z}) + 2i(u + v)\frac{1}{2}(z + \frac{1}{z}) - \sqrt{2}i(u - v)\frac{1}{2i}(z - \frac{1}{z}) - \sqrt{3}} \frac{dz}{z} \\
&= \frac{\sqrt{2}i}{\pi} \int_C \frac{1}{(z^2 + 1) + 2i(u + v)(z^2 + 1) - \sqrt{2}(u - v)(z^2 - 1) - 2\sqrt{3}z} dz \\
&= \frac{\sqrt{2}i}{\pi} \int_C \frac{1}{(1 - \sqrt{2}u + \sqrt{2}v + 2iu + 2iv)z^2 - 2\sqrt{3}z + (1 + \sqrt{2}u - \sqrt{2}v + 2iu + 2iv)} dz, \\
G(u, v) &= -\frac{2i}{\sqrt{6}\pi} \int_C \frac{1}{\left[\frac{1}{2}(z + \frac{1}{z}) + 2i(u + v)\frac{1}{2}(z + \frac{1}{z}) - \sqrt{2}i(u - v)\frac{1}{2i}(z - \frac{1}{z}) - \sqrt{3} \right]^2} \frac{dz}{z} \\
&= -\frac{8i}{\sqrt{6}\pi} \int_C \frac{z}{\left[(z^2 + 1) + 2i(u + v)(z^2 + 1) - \sqrt{2}(u - v)(z^2 - 1) - 2\sqrt{3}z \right]^2} dz \\
&= -\frac{8i}{\sqrt{6}\pi} \int_C \frac{z}{\left[(1 - \sqrt{2}u + \sqrt{2}v + 2iu + 2iv)z^2 - 2\sqrt{3}z + (1 + \sqrt{2}u - \sqrt{2}v + 2iu + 2iv) \right]^2} dz.
\end{aligned}$$

The two poles $z_{\text{pos}}, z_{\text{neg}}$ of each integrand are

$$\begin{aligned}
&\frac{2\sqrt{3} \pm \sqrt{12 - 4(1 - \sqrt{2}u + \sqrt{2}v + 2iu + 2iv)(1 + \sqrt{2}u - \sqrt{2}v + 2iu + 2iv)}}{2(1 - \sqrt{2}u + \sqrt{2}v + 2iu + 2iv)} \\
&= \frac{\sqrt{3} \pm \sqrt{2}\sqrt{(u - i)(3u + i) + (v - i)(3v + i) + 2uv - 1}}{1 - \sqrt{2}u + \sqrt{2}v + 2iu + 2iv}
\end{aligned}$$

and z_{neg} is always inside C , z_{pos} is always outside. For F , z_{neg} is a pole of order 1 and the associated residue is

$$\begin{aligned} & \lim_{z \rightarrow z_{\text{neg}}} \frac{1}{(1 - \sqrt{2}u + \sqrt{2}v + 2iu + 2iv)(z - z_{\text{pos}})} \\ &= -\frac{1}{2\sqrt{2}} \frac{1}{\sqrt{(u-i)(3u+i) + (v-i)(3v+i) + 2uv - 1}}; \end{aligned}$$

multiplying by $(2\pi i)(\sqrt{2}i/\pi)$ completes the proof. For G , z_{neg} is a pole of order 2 and the associated residue is

$$\begin{aligned} & \lim_{z \rightarrow z_{\text{neg}}} \frac{d}{dz} \left\{ \frac{z}{(1 - \sqrt{2}u + \sqrt{2}v + 2iu + 2iv)^2(z - z_{\text{pos}})^2} \right\} \\ &= \frac{\sqrt{6}}{16} \frac{1}{[(u-i)(3u+i) + (v-i)(3v+i) + 2uv - 1]^{3/2}}; \end{aligned}$$

multiplying by $(2\pi i)(-8i/(\sqrt{6}\pi))$ completes the proof. Clearly there is common structure to both cases and a more encompassing theorem should be possible.

6.3. Pinned Case: Backward Direction. The fact that $G = F^3$ is fairly miraculous but completely unmotivated. Let us briefly sketch a “backward” argument for the pinned case only, starting with a known inverse Fourier transform [30]:

$$\int_{-\infty}^{\infty} \frac{\exp(-iv y)}{\sqrt{(u-i)^2 + (v-i)^2 + 3}} dv = 2 \exp(y) K_0 \left(|y| \sqrt{(u-i)^2 + 3} \right).$$

Differentiating both sides with respect to u , we obtain

$$-\int_{-\infty}^{\infty} \frac{\exp(-iv y)(u-i)}{[(u-i)^2 + (v-i)^2 + 3]^{3/2}} dv = -2 \exp(y) K_1 \left(|y| \sqrt{(u-i)^2 + 3} \right) \cdot \frac{|y|(u-i)}{\sqrt{(u-i)^2 + 3}}$$

that is,

$$\int_{-\infty}^{\infty} \frac{\exp(-iv y)}{[(u-i)^2 + (v-i)^2 + 3]^{3/2}} dv = 2|y| \exp(y) \frac{K_1 \left(|y| \sqrt{(u-i)^2 + 3} \right)}{\sqrt{(u-i)^2 + 3}}.$$

Another known inverse Fourier transform is useful now [30]:

$$\int_{-\infty}^{\infty} \exp(-iu x) \frac{K_1 \left(|y| \sqrt{(u-i)^2 + 3} \right)}{\sqrt{(u-i)^2 + 3}} du$$

$$\begin{aligned}
&= \frac{\sqrt{2\pi} \exp(x)}{3^{1/4} |y|} (x^2 + y^2)^{1/4} K_{1/2} \left(\sqrt{3} \sqrt{x^2 + y^2} \right) \\
&= \frac{\sqrt{2\pi} \exp(x)}{3^{1/4} |y|} (x^2 + y^2)^{1/4} \sqrt{\frac{\pi}{2}} \frac{\exp \left(-\sqrt{3} \sqrt{x^2 + y^2} \right)}{3^{1/4} (x^2 + y^2)^{1/4}} \\
&= \frac{\pi}{\sqrt{3}} \frac{\exp(x)}{|y|} \exp \left(-\sqrt{3} \sqrt{x^2 + y^2} \right).
\end{aligned}$$

Multiplying both sides by $2|y| \exp(y)/(2\pi)^2$, we conclude that

$$\frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\exp(-iu x - iv y)}{[(u-i)^2 + (v-i)^2 + 3]^{3/2}} du dv = \frac{1}{2\sqrt{3}\pi} \exp \left(x + y - \sqrt{3} \sqrt{x^2 + y^2} \right)$$

as was to be shown.

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